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Highlights

- The paper deals with uncertain convex multiobjective problems.
- We consider ball uncertainty affecting all data.
- We define a radius of highly robust weak efficiency certifying its existence.
- We provide bounds, and an exact formula, for this radius.
- We provide simple formulas for convex quadratic and linear multi-objective programs.
- These formulas are applied to two variants of a test problem due to Ben-Tal and Nemirovski.

Guaranteeing highly robust weakly efficient solutions for uncertain multi-objective convex programs*

M.A. Goberna^{†‡} V. Jeyakumar[§] G. Li[§] J. Vicente-Pérez[¶]

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Abstract

This paper deals with uncertain multi-objective convex programming problems, where the data of the objective function or the constraints or both are allowed to be uncertain within specified uncertainty sets. We present sufficient conditions for the existence of highly robust weakly efficient solutions, that is, robust feasible solutions which are weakly efficient for any possible instance of the objective function within a specified uncertainty set. This is done by way of estimating the radius of highly robust weak efficiency under linearly distributed uncertainty of the objective functions. In the particular case of robust quadratic multi-objective programs, we show that these sufficient conditions can be expressed in terms of the original data of the problem, extending and improving the corresponding results in the literature for robust multi-objective linear programs under ball uncertainty.

Keywords. Robustness and sensitivity analysis. Multi-objective optimization. Convex optimization. Robust optimization. Robust efficient solutions.

1 Introduction

Many decision-making problems arising in practice can be modelled as multi-objective optimization programs where some of the data of the constraint functions or the objective functions or both are uncertain. The robust optimization approach (see, e.g., [2]) provides a deterministic framework for studying such uncertain decision-making problems by assuming that all data depend on parameters ranging on prescribed sets, called uncertainty sets, whose elements are called scenarios.

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[†]Corresponding author. Tel.: +34 965903533. Fax: +34 965903531.

[‡]Departamento de Matemáticas, Universidad de Alicante, 03071 Alicante, Spain.

[§]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia.

[¶]Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, 03071 Alicante, Spain.

E-mail addresses: mgoberna@ua.es (M.A. Goberna), v.jeyakumar@unsw.edu.au (V. Jeyakumar), g.li@unsw.edu.au (G. Li), jose.vicente@ua.es (J. Vicente-Pérez).

For the sake of simplicity, we assume that the data have independent uncertainty sets, which are singletons when the data are deterministic. The robust feasible solutions are those elements of the decision space \mathbb{R}^n which are feasible for any conceivable instance (that is, for any realization of the uncertainty in the given uncertainty sets) and the robust feasible set is the set of all robust feasible solutions.

A common objection to the robust optimization approach to uncertain decision-making problems is based on the fact that one may have an empty robust feasible set. For instance, if the uncertainty set is a ball and if the radius of the uncertainty set is too large then the robust feasible set may be empty. Recent research has addressed the issue of guaranteeing robust feasibility by providing formulas for the *radius of robust feasibility* for uncertain optimization problems with finitely many linear constraints ([9], [17]), finitely many convex constraints [15], and infinitely many linear constraints [16].

Computing the radius of robust feasibility, for example, for linearly constrained problems, is equivalent to computing the distance to ill-posedness in the sense of stability analysis ([7], [8]), that is, the supremum of the size of those perturbations of the data of a nominal problem which preserve its structure and feasibility. The formulas for the radius of robust feasibility are expressed as the distance from certain closed convex subsets of \mathbb{R}^{n+1} to the origin. As shown in [9], [15] and [17], these distances can be computed by solving numerically tractable optimization problems (e.g., linearly constrained convex quadratic programs, second-order cone programs or semi-definite programs) in particular cases. These results were obtained assuming that the data uncertainty is only present in the constraints.

On the other hand, robust solutions of uncertain multi-objective linear optimization in the face of data uncertainty in the objective function in the case of interval uncertainty sets were studied under the name of *necessary efficient solution* by Bitran [5], Inuiguchi and Sakawa [22], Oliveira and Antunes [29], Hladík [19] and other references therein. The related notion of *highly robust solution* for multi-objective optimization was introduced by Ide and Schöbel [21] and further studied by Kuhn, Raith, Schmidt and Schöbel [23], and Dranichak and Wiecek [32], among others.

Many other concepts of robust solution for uncertain multi-objective optimization problems have been proposed recently (see the survey papers [21] and [32]). In particular, the concept of minimax robust solution for multi-objective uncertain optimization was given by Kuroiwa and Lee [24] in 2012, who defined the robust counterpart of a given uncertain optimization problem with m objective functions f_i , $i = 1, \dots, m$, as the deterministic optimization problem consisting in the simultaneous minimization of the worst-case functions $\sup_{u_i \in \mathcal{U}_i} f_i(x, u_i)$ on the robust feasible set X , where \mathcal{U}_i denotes the uncertainty set of the objective functions f_i , $i = 1, \dots, m$. The optimal solutions, in the sense of multi-objective optimization, of the robust counterpart problem are called minimax robust solutions. Comparative studies on robust solutions for uncertain multi-objective optimization can be found in [11], [20] and [21, Section 3].

As an illustration, consider, for instance, the portfolio management problem under data uncertainty, where the decision maker may invest C euros in a portfolio comprised of n assets (shares, stocks, securities). The decision variables are the amount of euros to be invested in the i -th asset, denoted by x_i , $i = 1, \dots, n$.

Any portfolio $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n x_i \leq C$, and other (possibly uncertain) linear constraints $a_j^\top x + b_j \leq 0$, $j \in J$ (where J is some index set, the superscript $^\top$ means transpose and so, $a_j^\top x$ expresses the standard inner product $\langle a_j, x \rangle$ of $a_j, x \in \mathbb{R}^n$). Let $\bar{a}_j \in \mathbb{R}^n$ and $\bar{b}_j \in \mathbb{R}$ be estimations of a_j and b_j , and let $\mathcal{V}_j \subseteq \mathbb{R}^{n+1}$ with $(\bar{a}_j, \bar{b}_j) \in \mathcal{V}_j$ be the uncertainty set for (a_j, b_j) , $j \in J$. Thus, the robust feasible set is

$$X := \left\{ x \in \mathbb{R}^n : \begin{array}{l} a_j^\top x + b_j \leq 0, \forall (a_j, b_j) \in \mathcal{V}_j, \forall j \in J, \\ \sum_{i=1}^n x_i \leq C, x_i \geq 0, i = 1, \dots, n \end{array} \right\},$$

which is a polyhedral convex set whenever J is finite and the convex hull of \mathcal{V}_j is a polytope (e.g., when \mathcal{V}_j is finite) for all $j \in J$. Assume that $X \neq \emptyset$. Let r_i be the return per 1 euro invested in asset i during a period of time. Since these returns are not known in advance, $r = (r_1, \dots, r_n)$ is an uncertain vector.

The portfolio problem under uncertainty consists in the simultaneous maximization of the expected return $r^\top x$ and minimization of its variance (interpreted as volatility), say $x^\top \bar{A} x$. Assume that the vector $\bar{r} \in \mathbb{R}_+^n$ and the positive semi-definite matrix \bar{A} are the estimated mean and covariance matrix of the return vector, respectively. Then, the nominal problem is

$$\begin{aligned} (\bar{P}) \quad & \text{V-min} \quad (-\bar{r}^\top x, x^\top \bar{A} x) \\ & \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j \leq 0, j \in J, \\ & \quad \sum_{i=1}^n x_i \leq C, x_i \geq 0, i = 1, \dots, n, \end{aligned}$$

where V-min stands for *vector minimization*.

Let $\mathcal{U}_1 \subseteq \mathbb{R}^n$ and \mathcal{U}_2 (a family of symmetric matrices in $\mathbb{R}^{n \times n}$) be the uncertainty sets for r and A , respectively. In this case, it is reasonable to assume that the uncertainty is “objective-wise” in the sense that $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$. Then, $\bar{x} \in X$ is a *highly robust solution* when it is a solution of

$$(\bar{P}_{(r,A)}) \quad \text{V-min}_{x \in X} \quad (-r^\top x, x^\top A x)$$

for all $(r, A) \in \mathcal{U}$, and it is a *minmax robust solution* when it is an optimal solution of the robust counterpart

$$(\hat{P}) \quad \text{V-min}_{x \in X} \quad \left(-\inf_{r \in \mathcal{U}_1} r^\top x, \sup_{A \in \mathcal{U}_2} x^\top A x \right).$$

The existence of highly robust solutions is also clearly dependent on the size of the uncertainty sets for the objective functions, so that it is important to get formulas for the radius of highly robustness guaranteeing the existence of this class of solutions. This problem was initially tackled by Georgiev, Luc and Pardalos [12] and by the authors [17] in the linear setting.

This paper goes further by considering convex (in particular quadratic) multi-objective uncertain programs. It is worth observing that the results provided in this paper are new even in the robust scalar optimization setting. We make the following contributions to robust multi-objective optimization:

- In Section 3, we establish lower and upper estimates for the radius of highly robust weak efficiency under general case of uncertainty. In the special case where the objective

functions are affected by the commonly used ball uncertainty, we obtain an exact formula for the radius. We employ the idea of sharp efficiency [13] and its corresponding sharpness modulus to obtain these estimates.

- In Section 4, as an application of the radius estimates, we present sufficient conditions for the non-emptiness of the set of highly robust weakly efficient solutions. We also provide conditions characterizing highly robust weakly efficient solutions.
- In Section 5, we demonstrate that, in the particular case of uncertain convex quadratic multi-objective programs, the obtained sufficient conditions for existence of highly robust weak efficiency can be expressed in terms of the original data of the problem, extending and improving results in [16, 17] for robust multi-objective linear programs under ball uncertainty.

2 Preliminaries: highly robust efficient solutions

In this section, we present the notion of highly robust weak efficiency and its connection with the known concept of minmax robust efficiency. We begin by recalling some basic definitions and notation which will be used later on.

By convention, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. We denote by 0_n , $\|\cdot\|$, $\|\cdot\|_1$, \mathbb{R}_+^n and \mathbb{B}_n , the vector of zeros, the Euclidean norm, the L_1 norm, the non-negative orthant and the closed unit ball in \mathbb{R}^n , respectively. The *simplex* in the space of criteria \mathbb{R}^m is defined by $\Delta_m := \{\lambda \in \mathbb{R}_+^m : \sum_{i=1}^m \lambda_i = 1\}$. Given $Z \subseteq \mathbb{R}^n$, $\text{int } Z$, $\text{bd } Z$ and $\text{conv } Z$, denote the interior, the boundary and the convex hull of Z , respectively, whereas $\text{cone } Z := \mathbb{R}_+ \text{conv } Z$ denotes the convex conical hull of $Z \cup \{0_n\}$. By convention, the product of the scalar 0 by any set $Z \subseteq \mathbb{R}^n$ is $\{0_n\}$. The orthogonal complement of a linear subspace Z is denoted by Z^\perp .

The *negative polar* of a convex cone $K \subseteq \mathbb{R}^n$ is $K^\circ := \{w \in \mathbb{R}^n : w^\top x \leq 0 \ \forall x \in K\}$. Additionally, if Z is a convex subset of \mathbb{R}^n and $\bar{z} \in Z$, the *cone of feasible directions* of Z at \bar{z} is

$$D(Z, \bar{z}) := \{v \in \mathbb{R}^n : \bar{z} + \alpha v \in Z \text{ for some } \alpha > 0\} = \mathbb{R}_+ (Z - \bar{z})$$

while the *normal cone* of Z at $\bar{z} \in Z$ is the negative polar of $D(Z, \bar{z})$, that is,

$$N(Z, \bar{z}) := \{w \in \mathbb{R}^n : w^\top (z - \bar{z}) \leq 0 \ \forall z \in Z\}.$$

Moreover, for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, its *subdifferential* at \bar{x} is defined by

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n : v^\top (x - \bar{x}) \leq f(x) - f(\bar{x}) \ \forall x \in \mathbb{R}^n\}.$$

In this paper we consider deterministic convex multi-objective programs of the form

$$\begin{aligned} (\bar{P}) \quad & \text{V-min} \quad (\bar{f}_1(x), \dots, \bar{f}_m(x)) \\ & \text{s.t.} \quad \bar{g}_j(x) \leq 0, \ j = 1, \dots, p, \end{aligned} \tag{1}$$

where $\bar{g}_j : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper convex lower semicontinuous extended functions for all $j \in J := \{1, \dots, p\}$, and $\bar{f}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex functions for all $i \in I := \{1, \dots, m\}$. Note that we confine ourselves with real-valued objective functions so that the space of criteria

\mathbb{R}^m does not need to be completed with a point at infinity and the concepts of solutions do not have to be replaced by the more involved ones of vector-valued analysis (see, e.g., [6, 28] and references therein). The assumptions on the constraint functions guarantee the closedness and the convexity of the feasible set \bar{X} of the *nominal problem* (\bar{P}) .

In most applications of uncertain multi-objective optimization, conflicting objective functions depend on different uncertainty sets. Thus, we adopt the so-called *objective-wise* assumption, that is, we assume that the uncertainty set \mathcal{U} can be expressed as $\prod_{i=1}^m \mathcal{U}_i$. This way, the multi-objective problem (\bar{P}) in (1), in the face of data uncertainty in the constraints and linearly distributed data uncertainty in the objective functions, can be captured by a parameterized multi-objective problem

$$(P_{u,v})_{u \in \mathcal{U}, v \in \mathcal{V}} \quad \begin{array}{ll} \text{V-min} & (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle) \\ \text{s.t.} & g_j(x, v_j) \leq 0, j \in J, \end{array} \quad (2)$$

referred to simply as $(P_{u,v})$, where $g_j(\cdot, v_j)$ is a proper convex lower semicontinuous extended real-valued function for all $v_j \in \mathcal{V}_j$, $j \in J$, and the parameters $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_p)$ lie in the uncertainty sets $\mathcal{U} := \prod_{i=1}^m \mathcal{U}_i$ and $\mathcal{V} := \prod_{j=1}^p \mathcal{V}_j$.

Following the robust optimization approach, by enforcing the constraints for all possible uncertainties within \mathcal{V}_j , $j \in J$, $(P_{u,v})$ is replaced then by the *single parametric problem*

$$(P_u)_{u \in \mathcal{U}} \quad \begin{array}{ll} \text{V-min} & (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle) \\ \text{s.t.} & g_j(x, v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j \in J, \end{array}$$

whose feasible set,

$$X := \{x \in \mathbb{R}^n : g_j(x, v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j \in J\} \quad (3)$$

is called *robust feasible set* of $(P_{u,v})$ because its elements are feasible solutions of $(P_{u,v})$ for any conceivable scenario. In short, we shall denote $f_i(x, u_i) := \bar{f}_i(x) + \langle u_i, x \rangle$ for all $u_i \in \mathcal{U}_i$, $i \in I$. Observe that X is independent of $u \in \mathcal{U}$. From now on we assume that the robust feasible set X is non-empty.

In this paper, we examine the concept of highly robust weakly efficient solution (see Definition 1 below) in the framework of uncertain multi-objective convex programming. It is easy to see from the definition that the set of highly robust weakly efficient solutions may be empty. The non-emptiness of this solution set depends on the size of the uncertainty sets for the objectives (see also Example 11 below for a specific instance). So, developing conditions that guarantee the existence of highly robust weakly efficient solutions has emerged as a critical question in the area of robust multi-objective optimization.

Definition 1 (Highly robust weakly efficient solution) We say that $\bar{x} \in X$ is a highly robust weakly efficient solution for $(P_{u,v})$ if, for each $u \in \mathcal{U}$, \bar{x} is a weakly efficient solution to (P_u) , that is, if, for each $u \in \mathcal{U}$, there exists no $x \in X$ such that $f_i(x, u_i) < f_i(\bar{x}, u_i)$ for all $i \in I$.

We denote by X_h the set of highly robust weakly efficient solutions of $(P_{u,v})$. According to the well-known characterization of the weakly efficient solutions of multi-objective programs via scalarization, the set of highly robust weakly efficient solutions can be expressed as

$$X_h = \bigcap_{u \in \mathcal{U}} \bigcup_{\lambda \in \Delta_m} \operatorname{argmin} \{ \lambda^\top f(x, u) : x \in X \},$$

where $\lambda^\top f(\cdot, u) := \sum_{i=1}^m \lambda_i f_i(\cdot, u_i)$ for each $\lambda \in \Delta_m$, $u \in \mathcal{U}$.

Links to the known notion of minmax robust efficiency. In the literature, a popular notion of robust efficient solution for convex multi-objective program is the so-called minmax robust efficient solution [24, 25, 26, 27] which is defined as follows: $\bar{x} \in X$ is called a *minmax robust weakly efficient solution* to $(P_{u,v})$ if it is a weakly efficient solution to its robust counterpart (\hat{P}) given by

$$(\hat{P}) \quad \begin{array}{ll} \text{V-min} & \varphi(x) := (\varphi_1(x), \dots, \varphi_m(x)) \\ \text{s.t.} & x \in X, \end{array} \quad (4)$$

where $\varphi_i(x) := \sup_{u_i \in \mathcal{U}_i} f_i(x, u_i)$, that is, if there is no $x \in X$ such that $\varphi_i(x) < \varphi_i(\bar{x})$ for all $i \in I$. In other words, a robust feasible solution $\bar{x} \in X$ is a minmax robust solution for $(P_{u,v})$ when it is an optimal solution to the deterministic problem (\hat{P}) in (4), which is equivalent to the convex multi-objective program with linear objectives

$$\begin{array}{ll} \text{V-min} & (z_1, \dots, z_m) \\ \text{s.t.} & \varphi_i(x) - z_i \leq 0, i \in I, \\ & x \in X, \end{array} \quad (5)$$

in the sense that both problems are simultaneously feasible or not and that a point $\bar{x} \in X$ is a weakly efficient solution to (\hat{P}) if and only if $(\bar{x}, \varphi(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}^m$ is a weakly efficient solution to (5). As shown in [21], any highly robust weakly efficient solution to $(P_{u,v})$ is also a minmax robust weakly efficient solution.

3 Bounds for radius of highly robust weak efficiency

In this section, we consider an uncertain multi-objective programming problem with affine parametrization, that is, a problem as $(P_{u,v})$ in (2) where, for each $i \in I = \{1, \dots, m\}$, the objective function $\bar{f}_i(\cdot) + \langle u_i, \cdot \rangle$ is the result of perturbing the objective function \bar{f}_i of the nominal problem (\bar{P}) with the linear function $\langle u_i, \cdot \rangle$ for each $u_i \in \mathcal{U}_i := \beta \mathcal{W}_i$ with $\beta \geq 0$. Here, for each $i \in I$, the set \mathcal{W}_i is assumed to be convex compact with $0_n \in \text{int } \mathcal{W}_i$.

For each fixed $\beta \geq 0$, the uncertain problem can be captured by

$$(P_{u,v}^\beta)_{u \in \prod_{i=1}^m \beta \mathcal{W}_i, v \in \prod_{j=1}^p \mathcal{V}_j} \quad \begin{array}{ll} \text{V-min} & (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle) \\ \text{s.t.} & g_j(x, v_j) \leq 0, j \in J = \{1, \dots, p\}, \end{array}$$

and its corresponding single parametric problem has the form

$$(P_u^\beta)_{u \in \prod_{i=1}^m \beta \mathcal{W}_i} \quad \begin{array}{ll} \text{V-min} & (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle) \\ \text{s.t.} & g_j(x, v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j \in J. \end{array}$$

We denote by X_h^β the set of highly robust weakly efficient solutions of $(P_{u,v}^\beta)$. Observe that X_h^β shrinks when β is increased.

We also introduce the following deterministic multi-objective problem

$$(\tilde{P}) \quad \begin{array}{ll} \text{V-min} & (\bar{f}_1(x), \dots, \bar{f}_m(x)) \\ \text{s.t.} & g_j(x, v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j \in J, \end{array} \quad (6)$$

where the constraints are immunized against the data uncertainty, but the objective data uncertainty has not been taken into account. For this reason, we refer (\tilde{P}) as the *semi-robust counterpart* of $(P_{u,v}^\beta)$. Notice that (P_u^β) with $\beta = 0$ coincides with the semi-robust counterpart problem (\tilde{P}) .

Definition 2 (Radius of highly robust weak efficiency) We define the radius of highly robust weak efficiency of problem (\tilde{P}) in (6) as

$$\delta(\tilde{P}) := \sup \left\{ \beta \in \mathbb{R}_+ : X_h^\beta \neq \emptyset \right\}.$$

Note that, in the interest of simplicity, the uncertainty sets \mathcal{U}_i , for each $i \in I$, are defined as $\mathcal{U}_i = \beta \mathcal{W}_i$ instead of $\mathcal{U}_i = \beta_i \mathcal{W}_i$. See Remark 6 later in this section for details.

Links to other notions of radius of robustness. Another concept of radius of robustness has been introduced in [12, Section 3], in the framework of uncertain linear multi-objective optimization under linear perturbations of the objective function, as the size, $r(\bar{x})$, of the greatest perturbations which preserve the efficiency of a given feasible solution \bar{x} . So, the differences between both definitions are that one is local while the other is global and that one is referred to efficiency and the other one to weak efficiency. Since any efficient solution is weakly efficient, it is obvious that $r(\bar{x}) \leq \delta(\tilde{P})$ provided the size of the perturbations are measured in the same way. The formula for $r(\bar{x})$ in [12, Theorem 4.3] has been extended to nonlinear multi-objective optimization, always under linear perturbations, in [33, Theorem 4.2].

Basic properties of the radius of highly robust weak efficiency. We list here some basic properties for the radius of highly robust efficiency $\delta(\tilde{P})$.

- (i) Obviously, $\delta(\tilde{P}) = -\infty$ if and only if $X_h^0 = \emptyset$, that is, if (\tilde{P}) has no weakly efficient solution.
- (ii) If X is a singleton, then $\delta(\tilde{P}) = +\infty$. Moreover, the converse statement holds if X is bounded. To see this, let $\delta(\tilde{P}) = +\infty$ and let the robust feasible set X be bounded. We proceed by the method of contradiction and assume that X is not a singleton. Fix any $\bar{x} \in X$. Hence, there exists $\hat{x} \in X$ such that $\bar{x} \neq \hat{x}$. Since each \mathcal{W}_i is a compact convex set with nonempty interior, there exist $w_i \in \mathcal{W}_i$ and $\varepsilon > 0$ such that, for all $i \in I$,

$$w_i^\top (\bar{x} - \hat{x}) = \sup \{ u^\top (\bar{x} - \hat{x}) : u \in \mathcal{W}_i \} \geq \varepsilon \|\bar{x} - \hat{x}\| > 0.$$

Then, as X is bounded, there exists $L > 0$ such that for all $x, x' \in X$,

$$\max_{i \in I} \{ \bar{f}_i(x) - \bar{f}_i(x') \} \leq L \|x - x'\|.$$

So, for $\gamma > \frac{L}{\varepsilon}$, one has $\max_{i \in I} \{ \bar{f}_i(\hat{x}) - \bar{f}_i(\bar{x}) \} < \gamma w_i^\top (\bar{x} - \hat{x})$, that is,

$$\bar{f}_i(\hat{x}) + \gamma w_i^\top \hat{x} < \bar{f}_i(\bar{x}) + \gamma w_i^\top \bar{x} \quad \forall i \in I.$$

So, $\bar{x} \notin X_h^\gamma$ for all $\gamma > \frac{L}{\varepsilon}$. As \bar{x} is any arbitrary element in X , this implies that $X_h^\beta = \emptyset$ for all $\beta \geq 0$ with $\beta > \frac{L}{\varepsilon}$. In particular, we have $\delta(\tilde{P}) \leq \frac{L}{\varepsilon}$, which contradicts the fact that $\delta(\tilde{P}) = +\infty$. So, the conclusion follows.

- (iii) When at least one of the functions \bar{f}_i , $i \in I$, attains its minimum on X (e.g., when X is bounded or some \bar{f}_i is coercive on X), then there exists a highly robust weakly efficient solution to (\tilde{P}) , so that $\delta(\tilde{P}) \in \mathbb{R}_+ \cup \{+\infty\}$.

Now, we establish bounds for the radius of highly robust efficiency for problems under general uncertainty sets. To do this, let us introduce the concept of sharp efficient solution and its corresponding sharpness modulus for a deterministic multi-objective problem. These concepts will allow us to establish existence results for highly robust weakly efficient solutions of uncertain problems as well as formulas for computing the radius of highly robust efficiency.

Recall that X is the robust feasible set given as in (3). Let (\tilde{P}) be the semi-robust counterpart of $(P_{u,v}^\beta)$ defined in (6). We say an element $\bar{x} \in X$ is a *sharp efficient solution* for (\tilde{P}) if there exists a constant $k > 0$ such that

$$\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq k\|x - \bar{x}\|, \quad \forall x \in X. \quad (7)$$

The set of all sharp efficient solutions of (\tilde{P}) is denoted by $\mathcal{S}(\tilde{P})$. It easily follows from the definition that any sharp efficient solution for (\tilde{P}) is a weakly efficient solution for (\tilde{P}) . The *sharpness modulus* of $\bar{x} \in X$ regarding (\tilde{P}) is

$$\begin{aligned} \kappa(\bar{x}, \tilde{P}) &:= \sup \left\{ k \geq 0 : \max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq k\|x - \bar{x}\|, \quad \forall x \in X \right\} \\ &= \begin{cases} +\infty, & \text{if } X = \{\bar{x}\}, \\ \inf \left\{ \frac{\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\}}{\|x - \bar{x}\|} : x \in X \setminus \{\bar{x}\} \right\}, & \text{otherwise.} \end{cases} \end{aligned} \quad (8)$$

Direct verification shows that $\bar{x} \in \mathcal{S}(\tilde{P})$ if and only if $\kappa(\bar{x}, \tilde{P}) > 0$.

In passing, note that the notion of sharp efficient solution for a deterministic multi-objective problem like (\bar{P}) was introduced in [13] under the name of isolated efficient solution, and that the sharpness modulus is introduced here for the first time. The concept of sharp efficient solution extends the well-known one of isolated minimizer [1], previously called strong unique solution in [10] and sharp minimum in [30], from scalar to multi-objective programming.

We now establish bounds for the radius of robust efficiency for uncertain convex multi-objective programs under general uncertainty sets.

Theorem 3 (Bounds for radius: general uncertainty sets) *Let \mathcal{W}_i , $i \in I$, be compact convex sets such that $\varepsilon \mathbb{B}_n \subseteq \mathcal{W}_i \subseteq \rho \mathbb{B}_n$ with $0 < \varepsilon \leq \rho$. Assume that (\tilde{P}) has some weakly efficient solution. Then, for every $\beta \in \mathbb{R}$ with $0 < \beta < \delta(\tilde{P})$ one has*

$$\left\{ x \in \mathcal{S}(\tilde{P}) : \beta < \frac{\kappa(x, \tilde{P})}{\rho} \right\} \subseteq X_h^\beta \subseteq \left\{ x \in \mathcal{S}(\tilde{P}) : \beta \leq \frac{\kappa(x, \tilde{P})}{\varepsilon} \right\}. \quad (9)$$

Moreover, the following statements hold:

- (i) If $\mathcal{S}(\tilde{P}) \neq \emptyset$, then

$$\frac{\sup\{\kappa(x, \tilde{P}) : x \in \mathcal{S}(\tilde{P})\}}{\rho} \leq \delta(\tilde{P}) \leq \frac{\sup\{\kappa(x, \tilde{P}) : x \in \mathcal{S}(\tilde{P})\}}{\varepsilon}.$$

(ii) If $\mathcal{S}(\tilde{P}) = \emptyset$, then $\delta(\tilde{P}) = 0$.

Proof. We first note that, if X is a singleton, then the three sets in (9) coincide with X , and so, the conclusion trivially holds.

Let $\beta \in \mathbb{R}$ be such that $0 < \beta < \delta(\tilde{P})$. In order to prove the first inclusion in (9), let $\bar{x} \in \mathcal{S}(\tilde{P})$ be such that $\beta < \frac{\kappa(\bar{x}, \tilde{P})}{\rho}$. Take $k > 0$ be such that $\rho\beta < k < \kappa(\bar{x}, \tilde{P})$. Then, from (7) applied to (\tilde{P}) , given $x \in X$, one has $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq k\|x - \bar{x}\|$. So, if $u = (u_1, \dots, u_m) \in \prod_{i=1}^m \rho\beta\mathbb{B}_n$, then

$$\begin{aligned} \max_{i \in I} \{(\bar{f}_i(x) + \langle u_i, x \rangle) - (\bar{f}_i(\bar{x}) + \langle u_i, \bar{x} \rangle)\} &\geq \max_{i \in I} \{(\bar{f}_i(x) - \bar{f}_i(\bar{x})) - \|u_i\| \|x - \bar{x}\|\} \\ &\geq \max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x}) - \rho\beta \|x - \bar{x}\|\} \\ &\geq (k - \rho\beta) \|x - \bar{x}\|. \end{aligned}$$

Hence, \bar{x} turns out to be a sharp efficient solution (and so, weakly efficient solution) for any problem defined below with $u = (u_1, \dots, u_m)$ and $u_i \in \rho\beta\mathbb{B}_n$,

$$(P_u)_{u \in \prod_{i=1}^m \rho\beta\mathbb{B}_n} \quad \text{V-min}_{x \in X} \quad (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle).$$

Note that $\mathcal{W}_i \subseteq \rho\mathbb{B}_n$ for each $i \in I$. So, \bar{x} is a highly robust weakly efficient solution for (\tilde{P}) with uncertainty set $\mathcal{U}_i = \beta\mathcal{W}_i$, and hence $\bar{x} \in X_h^\beta$. Thus, the first inclusion in (9) holds.

Now, we establish the second inclusion in (9). Let $\bar{x} \in X_h^\beta$ where $0 < \beta < \delta(\tilde{P})$. Note that a highly robust weakly efficient solution for (\tilde{P}) with uncertainty set $\mathcal{U}_i = \beta\mathcal{W}_i$, is a weakly efficient solution of

$$\text{V-min}_{x \in X} \quad (\bar{f}_1(x) + \langle u_1, x \rangle, \dots, \bar{f}_m(x) + \langle u_m, x \rangle)$$

for all $u_i \in \beta\mathcal{W}_i$, $i \in I$. Since $\varepsilon\mathbb{B}_n \subseteq \mathcal{W}_i$ for all $i \in I$, then \bar{x} is a weakly efficient solution of $(P_u)_{u \in \prod_{i=1}^m \varepsilon\beta\mathbb{B}_n}$ for arbitrary vectors $u_i \in \varepsilon\beta\mathbb{B}_n$, $i \in I$. Denote $\gamma = \varepsilon\beta$. Take $\tilde{x} \in X$, $\tilde{x} \neq \bar{x}$, and $u_i := \gamma \frac{(\bar{x} - \tilde{x})}{\|\bar{x} - \tilde{x}\|} \in \varepsilon\beta\mathbb{B}_n$, $i \in I$. Using scalarization of weakly efficient solutions, there exists $\lambda \in \Delta_m$ such that

$$\bar{x} \in \operatorname{argmin} \left\{ (\lambda^\top \bar{f})(x) + \frac{\gamma}{\|\bar{x} - \tilde{x}\|} (\bar{x} - \tilde{x})^\top x : x \in X \right\}.$$

In particular,

$$(\lambda^\top \bar{f})(\bar{x}) + \frac{\gamma}{\|\bar{x} - \tilde{x}\|} (\bar{x} - \tilde{x})^\top \bar{x} \leq (\lambda^\top \bar{f})(\tilde{x}) + \frac{\gamma}{\|\bar{x} - \tilde{x}\|} (\bar{x} - \tilde{x})^\top \tilde{x},$$

so that

$$(\lambda^\top \bar{f})(\tilde{x}) - (\lambda^\top \bar{f})(\bar{x}) \geq \frac{\gamma}{\|\bar{x} - \tilde{x}\|} (\bar{x} - \tilde{x})^\top (\bar{x} - \tilde{x}) = \gamma \|\tilde{x} - \bar{x}\|. \quad (10)$$

Since $\lambda \in \Delta_m$, we also have

$$(\lambda^\top \bar{f})(\tilde{x}) - (\lambda^\top \bar{f})(\bar{x}) = \sum_{i=1}^m \lambda_i (\bar{f}_i(\tilde{x}) - \bar{f}_i(\bar{x})) \in \operatorname{conv} \{\bar{f}_i(\tilde{x}) - \bar{f}_i(\bar{x}) : i \in I\}$$

(an interval in \mathbb{R}), so that

$$(\lambda^\top \bar{f})(\tilde{x}) - (\lambda^\top \bar{f})(\bar{x}) \leq \max_{i \in I} \{\bar{f}_i(\tilde{x}) - \bar{f}_i(\bar{x})\}. \quad (11)$$

From (10) and (11) one gets $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \gamma \|x - \bar{x}\|$ for all $x \in X$, which shows that $\bar{x} \in \mathcal{S}(\tilde{P})$ and $\varepsilon\beta = \gamma \leq \kappa(\bar{x}, \tilde{P})$. So, $X_h^\beta \subseteq \left\{x \in \mathcal{S}(\tilde{P}) : \beta \leq \frac{\kappa(x, \tilde{P})}{\varepsilon}\right\}$.

To conclude the proof, it remains to show that $\delta(\tilde{P}) > 0$ if and only if $\mathcal{S}(\tilde{P}) \neq \emptyset$. Indeed, if $\delta(\tilde{P}) > 0$ and $0 < \beta < \delta(\tilde{P})$, then, in virtue of (9), there exists $\bar{x} \in X_h^\beta$ such that $\bar{x} \in \mathcal{S}(\tilde{P})$ and $\beta \leq \frac{\kappa(\bar{x}, \tilde{P})}{\varepsilon}$, which shows that $\mathcal{S}(\tilde{P}) \neq \emptyset$. Conversely, if $\mathcal{S}(\tilde{P}) \neq \emptyset$, taking an arbitrary $\bar{x} \in \mathcal{S}(\tilde{P})$, again by (9) one has $\bar{x} \in X_h^\beta$ for any $\beta < \frac{\kappa(\bar{x}, \tilde{P})}{\rho}$. This implies

$$\delta(\tilde{P}) \geq \frac{\kappa(\bar{x}, \tilde{P})}{\rho} > 0.$$

Therefore, the conclusion follows. \square

In the special case of ball uncertainty, we obtain an exact formula for the radius of highly robust weak efficiency in the next corollary.

Corollary 4 (Exact radius formula: ball uncertainty) *Consider the case for ball uncertainty sets, that is, $\mathcal{W}_i = \mathbb{B}_n$, $i \in I$. Assume that (\tilde{P}) has some weakly efficient solution. Then, for every $\beta \in \mathbb{R}$ with $0 < \beta < \delta(\tilde{P})$ one has*

$$\{x \in \mathcal{S}(\tilde{P}) : \beta < \kappa(x, \tilde{P})\} \subseteq X_h^\beta \subseteq \{x \in \mathcal{S}(\tilde{P}) : \beta \leq \kappa(x, \tilde{P})\}. \quad (12)$$

Moreover,

$$\delta(\tilde{P}) = \begin{cases} \sup\{\kappa(x, \tilde{P}) : x \in \mathcal{S}(\tilde{P})\}, & \text{if } \mathcal{S}(\tilde{P}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \quad (13)$$

and the supremum in (13) is attained at $\bar{x} \in \mathcal{S}(\tilde{P})$ if and only if $\bar{x} \in \bigcap_{0 < \beta < \delta(\tilde{P})} X_h^\beta$.

Proof. The inclusion (12) and the equality (13) follow by applying the preceding Theorem with $\varepsilon = \rho = 1$.

To see the last assertion, let $\bar{x} \in \mathcal{S}(\tilde{P})$ be such that $\delta(\tilde{P}) = \kappa(\bar{x}, \tilde{P})$. Then, by the first inclusion in (12) one has $\bar{x} \in \bigcap_{0 < \beta < \delta(\tilde{P})} X_h^\beta$. Conversely, if $\bar{x} \in \bigcap_{0 < \beta < \delta(\tilde{P})} X_h^\beta$, then the second inclusion in (12) yields $\beta \leq \kappa(\bar{x}, \tilde{P})$ for all β such that $0 < \beta < \delta(\tilde{P})$, that is, $\delta(\tilde{P}) \leq \kappa(\bar{x}, \tilde{P})$. This together with (13) yields $\delta(\tilde{P}) = \kappa(\bar{x}, \tilde{P})$. \square

Next, we illustrate Corollary 4 with an example.

Example 5 *Consider an uncertain bi-objective problem with corresponding single-parameterized problem*

$$(P_u^\beta)_{u \in \beta(\mathbb{B}_2 \times \mathbb{B}_2)} \quad \text{V-min}_{x \in X} \quad (x_1 + u_1^\top x, x_2 + u_2^\top x)$$

whose robust feasible set is $X = \text{conv} \{(1 - \cos t, 1 - \sin t) : t \in [0, \frac{\pi}{2}]\}$ and the uncertainty parameter u ranges on the uncertainty set $\mathcal{U} = \beta(\mathbb{B}_2 \times \mathbb{B}_2)$ whose size depends on the parameter $\beta \geq 0$, and $\bar{f}_i(x) = x_i$ for $i = 1, 2$. Its semi-robust counterpart (6) reads

$$(\tilde{P}) \quad V\text{-}\min_{x \in X} (x_1, x_2).$$

We now see that one can use Corollary 4 to identify the radius of highly robust weak efficiency. To do this, we first identify the set $\mathcal{S}(\tilde{P})$ of sharp efficient solutions of the deterministic bi-objective problem (\tilde{P}) . Let $x^t = (1 - \cos t, 1 - \sin t)$. Firstly, since any sharp efficient solution is weakly efficient, then $\mathcal{S}(\tilde{P}) \subseteq \{x^t : t \in [0, \frac{\pi}{2}]\}$ (the set of weakly efficient solutions).

Now we observe that $x^0 = (0, 1)$ is not a sharp efficient solution. Indeed, if there exists $k > 0$ such that $k\|(x_1, x_2 - 1)\| \leq \max\{x_1, x_2 - 1\} = x_1$ for all $x \in X$, we get $k^2 \leq \frac{x_1^2}{x_1^2 + (x_2 - 1)^2}$ for all $x \in X$, and hence $k^2 \leq \frac{(1 - \cos t)^2}{(1 - \cos t)^2 + (\sin t)^2}$ for all $t \in]0, \frac{\pi}{2}[$. By taking limits when $t \rightarrow 0$ we get $k^2 \leq 0$, which entails a contradiction. Following a similar reasoning, we get that $x^{\frac{\pi}{2}} = (1, 0)$ is not a sharp efficient solution either.

Consequently, we just have to check the sharp efficiency of $x^t = (1 - \cos t, 1 - \sin t)$ for $t \in]0, \frac{\pi}{2}[$. Thus, fix any $t \in]0, \frac{\pi}{2}[$. For each $i = 1, 2$, let

$$X_i := X \cap \{x \in \mathbb{R}^2 : x_{3-i} - x_{3-i}^t \leq x_i - x_i^t\},$$

$h_i(x) := \frac{(x_i - x_i^t)^2}{\|x - x^t\|^2} \in [0, 1]$ for $x \in D := \mathbb{R}^2 \setminus \{x^t\}$, and $\gamma_i := \inf\{h_i(x) : x \in X_i \setminus \{x^t\}\}$. According to (8), one has

$$\begin{aligned} \kappa^2(x^t, \tilde{P}) &= \inf_{x \in X \setminus \{x^t\}} \left\{ \left(\frac{\max\{x_1 - x_1^t, x_2 - x_2^t\}}{\|x - x^t\|} \right)^2 \right\} \\ &= \min \left\{ \inf_{x \in X_1 \setminus \{x^t\}} h_1(x), \inf_{x \in X_2 \setminus \{x^t\}} h_2(x) \right\} = \min\{\gamma_1, \gamma_2\}. \end{aligned}$$

The set of level $\alpha \in [0, 1]$ of h_i is given by

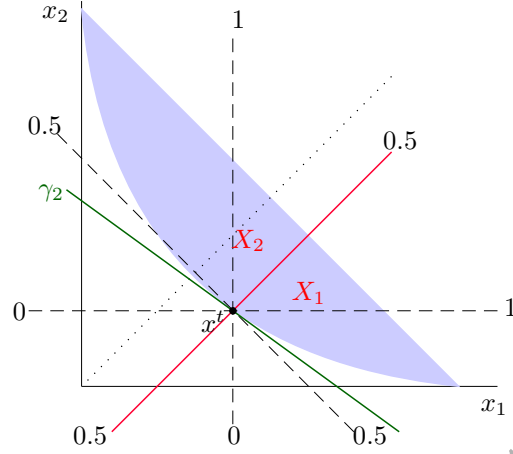
$$L(h_i, \alpha) = \begin{cases} \{x \in D : x_i = x_i^t\}, & \text{if } \alpha = 0, \\ \left\{ x \in D : x_{3-i} - x_{3-i}^t = \pm \left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{2}} (x_i - x_i^t) \right\}, & \text{else.} \end{cases}$$

Observe that $L(h_i, \alpha)$ is the intersection of D with a line when $\alpha = 0$ while it is the union of two lines, otherwise, all of them crossing x^t . Figure 1 represents X (the shadowed region), the line $x_2 - x_2^t = x_1 - x_1^t$ (the red line) which splits X into the sets X_1 and X_2 , and some level sets (the dashed lines) of h_1 (below the red line) and h_2 (above the red line) where each line is labeled with the corresponding level. In particular, $L(h_1, 0.5) = L(h_2, 0.5)$ is the union of two lines parallel to the bisectors of the four quadrants (the red line and its dashed orthogonal line).

In order to compute γ_1 and γ_2 we distinguish two cases:

- (i) If $t \geq \frac{\pi}{4}$ and so $x_1^t \geq x_2^t$ (as the situation illustrated in Figure 1), one has $\gamma_1 = \frac{1}{2}$ (attained at the points of X in the red line) and $\gamma_2 = (1 - x_1^t)^2$. Observe that γ_2

Figure 1: Illustration of Example 5.



is the level at which the corresponding level set of h_2 is tangent to the boundary of X at x^t (the green line in Figure 1). Since $x_1^t \geq 1 - \frac{\sqrt{2}}{2}$, one has $\gamma_2 \leq \frac{1}{2}$ and so $\kappa(x^t, \tilde{P}) = \sqrt{\gamma_2} = 1 - x_1^t$.

(ii) Analogously, if $t \leq \frac{\pi}{4}$, one gets $\kappa(x^t, \tilde{P}) = \sqrt{\gamma_1} = 1 - x_2^t$.

Therefore, every point $x^t = (1 - \cos t, 1 - \sin t)$, $t \in]0, \frac{\pi}{2}[$, is a sharp efficient solution for (\tilde{P}) with modulus $\kappa(x^t, \tilde{P}) = \min\{1 - x_1^t, 1 - x_2^t\} = \min\{\cos t, \sin t\}$. Thus, we have

$$\{\kappa(x, \tilde{P}) : x \in \mathcal{S}(\tilde{P})\} = \{\min\{\cos t, \sin t\} : t \in]0, \frac{\pi}{2}[\} =]0, \frac{\sqrt{2}}{2}],$$

whose supremum, $\delta(\tilde{P}) = \frac{\sqrt{2}}{2}$, is attained at $x^{\frac{\pi}{4}} = \left(1 - \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$ and $\bigcap_{0 < \beta < \frac{\sqrt{2}}{2}} X_h^\beta = \{x^{\frac{\pi}{4}}\}$

(the sets X_h^β are properly computed in Example 11 below). Observe that, for each $0 < \beta < \frac{\sqrt{2}}{2}$, we have

$$\begin{aligned} \{x \in \mathcal{S}(\tilde{P}) : \beta < \kappa(x, \tilde{P})\} &= \{x^t : t \in]\arcsin(\beta), \arccos(\beta)[\} \\ &\subsetneq \{x^t : t \in [\arcsin(\beta), \arccos(\beta)]\} = \{x \in \mathcal{S}(\tilde{P}) : \beta \leq \kappa(x, \tilde{P})\}, \end{aligned}$$

so that the lower and upper estimations of X_h^β in (12) do not coincide.

Remark 6 Recall that the notion of radius of highly robust weak efficiency has been defined in Definition 2 as $\delta(\tilde{P}) := \sup\{\beta \in \mathbb{R}_+ : X_h^\beta \neq \emptyset\}$ where the uncertainty sets are of the form $\mathcal{U}_i = \beta \mathcal{W}_i$, $i \in I$. On the other hand, if we consider $\mathcal{U}_i = \beta_i \mathcal{W}_i$ with $\beta_i \geq 0$, $i \in I$, then a plausible definition for the radius of highly robust weak efficiency (cf. [15]) is as follows:

$$\hat{\delta}(\tilde{P}) := \sup\{\min_{1 \leq i \leq m} \beta_i \in \mathbb{R}_+ : X_h^{(\beta_1, \dots, \beta_m)} \neq \emptyset\}.$$

In such a case, one can show $\hat{\delta}(\tilde{P}) = \delta(\tilde{P})$. Indeed, it is easy to see that

$$\hat{\delta}(\tilde{P}) \geq \sup\{\gamma \in \mathbb{R}_+ : X_h^{(\beta_1, \dots, \beta_m)} \neq \emptyset, \beta_i = \gamma, i \in I\} = \delta(\tilde{P}).$$

To see the equality, we suppose on the contrary that there exists $\alpha > 0$ be such that

$$\sup\{\min_{1 \leq i \leq m} \beta_i \in \mathbb{R}_+ : X_h^{(\beta_1, \dots, \beta_m)} \neq \emptyset\} > \alpha > \sup\{\gamma \in \mathbb{R}_+ : X_h^{(\beta_1, \dots, \beta_m)} \neq \emptyset, \beta_i = \gamma, i \in I\}.$$

Then, there exists $(\bar{\beta}_1, \dots, \bar{\beta}_m) \in \mathbb{R}_+^m$ such that $\bar{\gamma} := \min_{1 \leq i \leq m} \bar{\beta}_i > \alpha$ and $X_h^{(\bar{\beta}_1, \dots, \bar{\beta}_m)} \neq \emptyset$.

Consequently, $X_h^{(\bar{\gamma}, \dots, \bar{\gamma})} \neq \emptyset$ and so,

$$\sup\{\gamma \in \mathbb{R}_+ : X_h^{(\beta_1, \dots, \beta_m)} \neq \emptyset, \beta_i = \gamma, i = 1, \dots, m\} \geq \bar{\gamma} > \alpha.$$

This makes a contradiction and thus, the desired equality holds. Therefore, one can apply previous results so as to compute $\hat{\delta}(\tilde{P})$.

4 Existence and characterizations of solutions

In this section, we provide verifiable conditions guaranteeing non-emptiness of highly robust weakly efficient solution sets. As an immediate implication of Theorem 3, a sufficient condition ensuring the existence of a highly robust weakly efficient solution is $\kappa(\bar{x}, \tilde{P}) > 0$ for some $\bar{x} \in X$, where X is the robust feasible set. On the other hand, this condition might not be directly verifiable because computing the modulus of sharp efficient solution is not a trivial task in general. To this end, in the next proposition, we establish upper and lower estimates of the modulus of sharp efficient solutions. This paves the way for obtaining verifiable conditions of the existence of highly robust weakly efficient solutions for uncertain convex multi-objective programs.

Proposition 7 (Bounds for the modulus of sharp solution) *Consider the uncertain multi-objective program $(P_{u,v}^\beta)$ and its semi-robust counterpart (\tilde{P}) . Let $\bar{x} \in X$ be such that $\Delta(\bar{x}) := \{\lambda \in \Delta_m : 0_n \in \sum_{i=1}^m \lambda_i \partial f_i(\bar{x}) + N(X, \bar{x})\} \neq \emptyset$. Then, one has*

$$v_1^* \leq \kappa(\bar{x}, \tilde{P}) \leq v_2^*,$$

where

$$\begin{aligned} v_1^* &= \sup_{\lambda \in \Delta(\bar{x})} \inf \left\{ \left\| \sum_{i=1}^m \lambda_i w_i + y \right\| : y \in \text{bd } N(X, \bar{x}), w_i \in \partial \bar{f}_i(\bar{x}) \right\}, \\ v_2^* &= \inf \left\{ \|y\| : y \in \text{bd} \left(\bigcup_{\lambda \in \Delta_m} \sum_{i=1}^m \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x}) \right) \right\}. \end{aligned}$$

Proof. If the robust feasible set X is a singleton set, then $N(X, \bar{x}) = \mathbb{R}^n$ and so $\kappa(\bar{x}, \tilde{P}) = v_1^* = v_2^* = \inf \emptyset = +\infty$. Thus, we assume that X contains at least two points. To see that $\kappa(\bar{x}, \tilde{P}) \geq v_1^*$, we proceed by the method of contradiction and suppose that there exists $\alpha \geq 0$ such that

$$\sup_{\lambda \in \Delta(\bar{x})} \inf \left\{ \left\| \sum_{i=1}^m \lambda_i w_i + y \right\| : y \in \text{bd } N(X, \bar{x}), w_i \in \partial \bar{f}_i(\bar{x}) \right\} > \alpha > \kappa(\bar{x}, \tilde{P}).$$

Then, there exists $\lambda \in \Delta(\bar{x})$ such that, for all $y \in \text{bd } N(X, \bar{x})$ and $w_i \in \partial \bar{f}_i(\bar{x})$,

$$\left\| -\sum_{i=1}^m \lambda_i w_i - y \right\| > \alpha, \quad (14)$$

and there exists $x' \neq \bar{x}$ with $x' \in X$ such that

$$\max_{i \in I} \{\bar{f}_i(x') - \bar{f}_i(\bar{x})\} < \alpha \|x' - \bar{x}\|. \quad (15)$$

As $\lambda \in \Delta(\bar{x})$, it follows that there exist $\bar{w}_i \in \partial \bar{f}_i(\bar{x})$ such that $-\sum_{i=1}^m \lambda_i \bar{w}_i \in N(X, \bar{x})$. This together with the relation in (14) shows that

$$-\sum_{i=1}^m \lambda_i \bar{w}_i + \alpha \mathbb{B}_n \subseteq N(X, \bar{x}). \quad (16)$$

Now, let $w := -\alpha \frac{x' - \bar{x}}{\|x' - \bar{x}\|}$. Clearly, $\|w\| = \alpha$. Let h be the convex function defined by $h(x) := \left(\sum_{i=1}^m \lambda_i \bar{w}_i + w \right)^\top (x - \bar{x})$. Then, (16) implies that $0_n \in \nabla h(\bar{x}) + N(X, \bar{x})$ and so, \bar{x} is a minimizer of h on X . Thus,

$$\sum_{i=1}^m \lambda_i \bar{w}_i (x' - \bar{x}) - \alpha \|x' - \bar{x}\| = h(x') \geq h(\bar{x}) = 0.$$

This together with $\lambda \in \Delta_m$ and $\bar{w}_i \in \partial \bar{f}_i(\bar{x})$ implies that

$$\max_{i \in I} \{\bar{f}_i(x') - \bar{f}_i(\bar{x})\} \geq \sum_{i=1}^m \lambda_i (\bar{f}_i(x') - \bar{f}_i(\bar{x})) \geq \sum_{i=1}^m \lambda_i \bar{w}_i^\top (x' - \bar{x}) \geq \alpha \|x' - \bar{x}\|,$$

which contradicts (15). So, $\kappa(\bar{x}, \tilde{P}) \geq v_1^*$.

We now show that $\kappa(\bar{x}, \tilde{P}) \leq v_2^*$. Let $\gamma \geq 0$ be such that $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \gamma \|x - \bar{x}\|$ for all $x \in X$, and so $\kappa(\bar{x}, \tilde{P}) \geq \gamma$. Denote $F(x) := \max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\}$. By the Valadier formula (see, e.g., [18, Thm. 4.4.2]), one has

$$\partial F(\bar{x}) = \text{conv} \left(\bigcup_{i \in I(\bar{x})} \partial \bar{f}_i(\bar{x}) \right),$$

where $I(\bar{x}) := \{i \in I : \bar{f}_i(\bar{x}) = F(\bar{x})\}$. Fix any $w \in \mathbb{R}^n$ such that $\|w\| \leq \gamma$ and let H be the convex function defined by $H(x) := F(x) - w^\top (x - \bar{x})$. Then, for all $x \in X$,

$$H(x) - H(\bar{x}) = F(x) - w^\top (x - \bar{x}) \geq \gamma \|x - \bar{x}\| - w^\top (x - \bar{x}) \geq 0.$$

So, H attains its global minimum over X at \bar{x} . It follows that $0_n \in \partial H(\bar{x}) + N(X, \bar{x})$, and hence, there exists $\lambda \in \Delta_m$ such that $w \in \sum_{i=1}^m \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x})$. Note that w was chosen as an arbitrary element in $\gamma \mathbb{B}_n$. Since

$$\gamma \mathbb{B}_n \subseteq \bigcup_{\lambda \in \Delta_m} \sum_{i=1}^m \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x}),$$

one has

$$v_2^* = \inf \left\{ \|y\| : y \in \text{bd} \left(\bigcup_{\lambda \in \Delta_m} \sum_{i=1}^m \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x}) \right) \right\} \geq \gamma.$$

From the above inequality and the definition of $\kappa(\bar{x}, \tilde{P})$ it follows that $v_2^* \geq \kappa(\bar{x}, \tilde{P})$. \square

Next, we use an example to illustrate that the estimates for the modulus of sharpness in Proposition 7 can be tight.

Example 8 Consider the following uncertain bi-objective problem with corresponding single-parameterized problem

$$(EP_u)_{u \in \mathcal{U}} \quad \text{V-min}_{x \in X} \quad \left(\frac{1}{2}x_1^2 + u_1^\top x, \frac{1}{2}x_2^2 + u_2^\top x \right)$$

whose robust feasible set is $X = [1, 2]^2$, and the uncertainty parameters $u_1 \in \mathcal{U}_1 = [-1, 1] \times \{0\}$ and $u_2 \in \mathcal{U}_2 = \{0\} \times [-1, 1]$. Its semi-robust counterpart problem (6) reads

$$(\widetilde{EP}) \quad \text{V-min}_{x \in X} \quad \left(\frac{1}{2}x_1^2, \frac{1}{2}x_2^2 \right).$$

Consider the feasible point $\bar{x} = (1, 1) \in X$ and let $\bar{f}_i(x) := \frac{1}{2}x_i^2$ for $i = 1, 2$. Direct verification shows that $N(X, \bar{x}) = -\mathbb{R}_+^2$ and $\Delta(\bar{x}) = \Delta_2 = \{\lambda \in \mathbb{R}_+^2 : \lambda_1 + \lambda_2 = 1\}$, whose symmetry center $(\frac{1}{2}, \frac{1}{2})$ we denote by $\bar{\lambda}$. Thus, we see in Figure 2, where the red and the blue polygonal lines represent $\bar{\lambda} + \text{bd } N(X, \bar{x})$ and $\text{bd}(\Delta_2 + N(X, \bar{x}))$, respectively, that

$$\begin{aligned} v_1^* &= \sup_{\lambda \in \Delta(\bar{x})} \inf \left\{ \left\| \sum_{i=1}^2 \lambda_i w_i + y \right\| : y \in \text{bd } N(X, \bar{x}), w_i \in \partial \bar{f}_i(\bar{x}) \right\} \\ &= \sup_{\lambda \in \Delta_2} \inf \{ \|\bar{\lambda} + y\| : y \in \text{bd } N(X, \bar{x}) \} \\ &= \inf \{ \|\bar{\lambda} + y\| : y \in \text{bd } N(X, \bar{x}) \} = 1/2, \end{aligned}$$

and

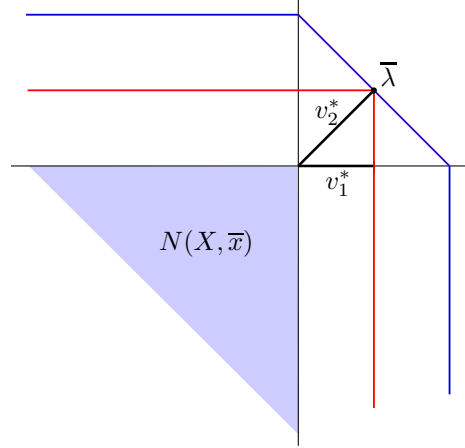
$$\begin{aligned} v_2^* &= \inf \left\{ \|y\| : y \in \text{bd} \left(\bigcup_{\lambda \in \Delta_2} \sum_{i=1}^2 \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x}) \right) \right\} \\ &= \inf \{ \|y\| : y \in \text{bd}(\Delta_2 + N(X, \bar{x})) \} = \|\bar{\lambda}\| = \sqrt{2}/2. \end{aligned}$$

Thus, Proposition 7 implies that $1/2 \leq \kappa(\bar{x}, \widetilde{EP}) \leq \sqrt{2}/2$.

Alternatively, simple calculations allow to verify that \bar{x} is a sharp efficient solution for (\widetilde{EP}) with $\kappa(\bar{x}, \widetilde{EP}) = \frac{\sqrt{2}}{2}$. Indeed, if $x = (x_1, x_2) \in X$ and $x_1 \geq x_2$, $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} = \frac{(x_1 - 1)(x_1 + 1)}{2} \geq x_1 - 1$ and $\|x - \bar{x}\| \leq \sqrt{2}(x_1 - 1)$, so that $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \frac{\sqrt{2}}{2} \|x - \bar{x}\|$. By symmetry, we have $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \frac{\sqrt{2}}{2} \|x - \bar{x}\|$ for all $x \in X$. This shows that $\kappa(\bar{x}, \widetilde{EP}) \geq \sqrt{2}/2$. To prove the equality $\kappa(\bar{x}, \widetilde{EP}) = \sqrt{2}/2$ we suppose, on the contrary, that there exists $\alpha > \sqrt{2}/2$ such that $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \alpha \|x - \bar{x}\|$ for all $x \in X$. We can assume that $\alpha < \sqrt{2}$ without loss of generality. Then, by taking $x = (\sqrt{2}\alpha, \sqrt{2}\alpha) \in X$, we see that

$$\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} = \frac{2\alpha^2 - 1}{2} \text{ and } \alpha \|x - \bar{x}\| = \alpha \sqrt{2}(\sqrt{2}\alpha - 1) = 2\alpha^2 - \sqrt{2}\alpha.$$

Figure 2: Geometric interpretation of the bounds for the modulus in Example 8.



Note that, as $\alpha > \sqrt{2}/2$,

$$\frac{2\alpha^2 - 1}{2} - (2\alpha^2 - \sqrt{2}\alpha) = -\alpha^2 + \sqrt{2}\alpha - \frac{1}{2} = \left(-\alpha - \frac{\sqrt{2}}{2}\right)^2 < 0$$

This contradicts the assumption that $\max_{i \in I} \{\bar{f}_i(x) - \bar{f}_i(\bar{x})\} \geq \alpha \|x - \bar{x}\|$ for all $x \in X$. So, $\kappa(\bar{x}, \widetilde{EP}) = \sqrt{2}/2$.

Theorem 9 (Sufficient conditions for existence of solutions) Let \mathcal{W}_i , $i \in I$, be compact convex sets with $0_n \in \text{int } \mathcal{W}_i$ such that $\mathcal{W}_i \subseteq \rho \mathbb{B}_n$ for some $\rho > 0$. Consider the uncertain multi-objective program $(P_{u,v})$ and its semi-robust counterpart (\tilde{P}) . Suppose that there exist $\bar{x} \in X$ and $\lambda \in \Delta_m$ such that $0_n \in \sum_{i=1}^m \lambda_i \partial \bar{f}_i(\bar{x}) + N(X, \bar{x})$ and

$$v_1^* = \inf \left\{ \left\| \sum_{i=1}^m \lambda_i w_i + y \right\| : y \in \text{bd } N(X, \bar{x}), w_i \in \partial \bar{f}_i(\bar{x}) \right\} > 0.$$

Then, $X_h^\beta \neq \emptyset$ for all $\beta \in [0, \frac{v_1^*}{\rho}]$.

Proof. From the assumptions, we see that v_1^* in Proposition 7 is positive. Thus, this Proposition guarantees that $\kappa(\bar{x}, \tilde{P}) \geq v_1^* > 0$. It then follows from Theorem 3 that $\delta(\tilde{P}) \geq \frac{v_1^*}{\rho} > 0$. Thus, the conclusion follows by the definition of radius of highly robust weak efficiency. \square

We now provide a complete characterization for highly robust weak efficiency in terms of the cones $G(x, u) := \text{cone}(\bigcup_{i=1}^m \partial_x f_i(x, u_i))$ for each $x \in X$, $u \in \mathcal{U}$. We note that the characterization below continues to hold for the general notion of highly robust weakly efficient solutions where the objective functions are subject to possibly nonlinear perturbations.

Proposition 10 (Characterizing highly robust weakly efficient solutions) *Assume that $\bar{x} \in X$ and $0_n \notin \bigcup_{u \in \mathcal{U}} \bigcup_{i=1}^m \partial_x f_i(\bar{x}, u_i)$. Then, \bar{x} is a highly robust weakly efficient solution if and only if*

$$D(X, \bar{x}) \cap \bigcup_{u \in \mathcal{U}} \text{int } G(\bar{x}, u)^\circ = \emptyset. \quad (17)$$

Proof. Since $0_n \notin \bigcup_{i=1}^m \partial_x f_i(\bar{x}, u_i)$ for each $u \in \mathcal{U}$, [14, Theorem 21] guarantees that \bar{x} is a weakly efficient solution to (P_u) if and only if $D(X, \bar{x}) \cap \text{int } G(\bar{x}, u)^\circ = \emptyset$. The conclusion follows from the definition of highly robust weakly efficient solutions. \square

Next, we illustrate Proposition 10 using the previous Example 5.

Example 11 *Consider the same example examined in 5, that is, the following uncertain bi-objective problem with corresponding single-parameterized problem*

$$(P_u^\beta)_{u \in \beta(\mathbb{B}_2 \times \mathbb{B}_2)} \quad \text{V-min}_{x \in X} (x_1 + u_1^\top x, x_2 + u_2^\top x)$$

whose robust feasible set is $X = \text{conv} \{(1 - \cos t, 1 - \sin t) : t \in [0, \frac{\pi}{2}]\}$ and the uncertainty parameter u ranges on the uncertainty set $\mathcal{U} = \beta(\mathbb{B}_2 \times \mathbb{B}_2)$, whose size depends on a size parameter $\beta \geq 0$. In this case, the deterministic problem in (6) reads

$$(\tilde{P}) \quad \text{V-min}_{x \in X} (x_1, x_2),$$

which can be seen as a particular instance of (P_u^β) by letting $\beta = 0$. Assume $0 < \beta < 1$. Since $\beta < 1$, one has

$$0_2 \notin \bigcup_{u \in \mathcal{U}} \{\nabla_x f_1(x, u_1), \nabla_x f_2(x, u_2)\} = \{(1, 0), (0, 1)\} + \beta \mathbb{B}_2 \quad \forall x \in X,$$

and we can apply Proposition 10 to compute the set X_h^β of highly robust weakly efficient solutions. Firstly, observe that any highly robust weakly efficient solution is, in particular, a weakly efficient solution to (\tilde{P}) , that is,

$$X_h^\beta \subseteq \{x^t : t \in [0, \frac{\pi}{2}]\},$$

where $x^t := (1 - \cos t, 1 - \sin t)$, $t \in [0, \frac{\pi}{2}]$. The points $x^0 = (0, 1)$ and $x^{\frac{\pi}{2}} = (1, 0)$, though, cannot be elements of X_h^β as they are no longer weakly efficient solutions to (P_u^β) for some arbitrarily small parameter u . For the rest of points of the arc, given $t \in]0, \frac{\pi}{2}[$, one has

$$D(X, x^t) = \{x \in \mathbb{R}^2 : x_2 > (-\cot t) x_1\} \cup \{0_2\},$$

with $-\cot t$ representing the slope of the tangent line to the arc at x^t . Moreover, if $0 < \beta \leq \frac{\sqrt{2}}{2}$, one gets

$$\bigcup_{u \in \mathcal{U}} \text{int } G(x^t, u)^\circ = \text{int cone}\{(-\sqrt{1 - \beta^2}, \beta), (\beta, -\sqrt{1 - \beta^2})\},$$

so that, (17) holds if and only if $\frac{\sqrt{1-\beta^2}}{-\beta} \leq -\cot t \leq \frac{-\beta}{\sqrt{1-\beta^2}}$ or, equivalently,

$$t \in T_h^\beta := [\arcsin(\beta), \arccos(\beta)].$$

We thus get

$$X_h^\beta = \begin{cases} \{x^t : t \in T_h^\beta\}, & \text{if } 0 \leq \beta \leq \frac{\sqrt{2}}{2}, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (18)$$

Hence, X_h^β is an arc of circle when $0 \leq \beta < \frac{\sqrt{2}}{2}$, a singleton when $\beta = \frac{\sqrt{2}}{2}$, and the empty set when $\beta > \frac{\sqrt{2}}{2}$.

5 Uncertain convex quadratic multi-objective programs

In this section, we examine the important class of convex quadratic multi-objective programs under uncertainty, and obtain sufficient conditions, expressed in terms of the original data of the problem, for the existence of highly robust weak efficiency.

Let \bar{A}_i be a given $n \times n$ symmetric positive semidefinite matrix, $\bar{c}_i, \bar{a}_j \in \mathbb{R}^n$ and $\bar{b}_j \in \mathbb{R}$ where $i \in I = \{1, \dots, m\}$ and $j \in J = \{1, \dots, p\}$. Let $\mathcal{W}_i \subseteq \mathbb{R}^n$, $i \in I$, be convex compact sets with $0_n \in \text{int } \mathcal{W}_i$, and $\mathcal{V}_j \subseteq \mathbb{R}^{n+1}$, $j \in J$, be convex compact sets, while $\beta \geq 0$. Consider the convex quadratic multi-objective programs under uncertainty:

$$(QP_{u,v}^\beta)_{u \in \prod_{i=1}^m \beta \mathcal{W}_i, (v, \gamma) \in \prod_{j=1}^p \mathcal{V}_j} \quad \text{V-min} \quad \left(\frac{1}{2}x^\top \bar{A}_1 x + \bar{c}_1^\top x + u_1^\top x, \dots, \frac{1}{2}x^\top \bar{A}_i x + \bar{c}_i^\top x + u_i^\top x \right) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \quad j \in J, \quad (19)$$

whose corresponding single parametric problem has the form

$$(QP_u^\beta)_{u \in \prod_{i=1}^m \beta \mathcal{W}_i} \quad \text{V-min} \quad \left(\frac{1}{2}x^\top \bar{A}_1 x + \bar{c}_1^\top x + u_1^\top x, \dots, \frac{1}{2}x^\top \bar{A}_i x + \bar{c}_i^\top x + u_i^\top x \right) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \quad \forall (v_j, \gamma_j) \in \mathcal{V}_j, \quad j \in J. \quad (20)$$

The corresponding semi-robust counterpart reads

$$(\widetilde{QP}) \quad \text{V-min} \quad \left(\frac{1}{2}x^\top \bar{A}_1 x + \bar{c}_1^\top x, \dots, \frac{1}{2}x^\top \bar{A}_i x + \bar{c}_i^\top x \right) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \quad \forall (v_j, \gamma_j) \in \mathcal{V}_j, \quad j \in J. \quad (21)$$

Throughout this section, we assume that the robust feasible set X , in this case given by $X = \{x : \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \quad \forall (v_j, \gamma_j) \in \mathcal{V}_j, \quad j \in J\}$, is a nonempty set.

Corollary 12 (Existence of solutions) *Let \mathcal{W}_i , $i \in I$, be compact convex sets with $0_n \in \text{int } \mathcal{W}_i$ such that $\mathcal{W}_i \subseteq \rho \mathbb{B}_n$ for some $\rho > 0$. Consider the uncertain quadratic multi-objective program $(QP_{u,v}^\beta)$ and its semi-robust counterpart (\widetilde{QP}) . Suppose that there exist $\bar{x} \in X$ and $\lambda \in \Delta_m$ such that $-\sum_{i=1}^m \lambda_i (\bar{A}_i \bar{x} + \bar{c}_i) \in N(X, \bar{x})$ and*

$$v_1^* = \inf \left\{ \left\| \sum_{i=1}^m \lambda_i (\bar{A}_i \bar{x} + \bar{c}_i) + y \right\| : y \in \text{bd } N(X, \bar{x}) \right\} > 0. \quad (22)$$

Then, a highly robust weakly efficient solution for $(QP_{u,v}^\beta)$ exists for all $\beta \in [0, \frac{v_1^}{\rho}]$.*

Proof. Let $\bar{f}_i(x) = \frac{1}{2}x^\top \bar{A}_i x + \bar{c}_i^\top x$, $i \in I$. Then, $\partial \bar{f}_i(\bar{x}) = \{\bar{A}_i \bar{x} + \bar{c}_i\}$, $i \in I$. Thus, the conclusion follows from Theorem 9. \square

It is worth noting that, in the case where \mathcal{V} is a polytope uncertainty set, the robust feasible set X is a polyhedral convex set and the boundary of normal cone of X at a particular point \bar{x} can be expressed as a finite union of polyhedral convex sets. In this case, the condition (22) can be checked by solving finitely many convex quadratic programs with linear constraints.

We now consider a linear multi-objective program under uncertainty

$$(LP_{u,v}^\beta)_{u \in \prod_{i=1}^m \mathcal{U}_i, v \in \prod_{j=1}^p \mathcal{V}_j} \quad \text{V-min} \quad (\bar{c}_1^\top x + u_1^\top x, \dots, \bar{c}_m^\top x + u_m^\top x) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, j \in J,$$

with uncertain objective functions $f_i(x, u_i) = \bar{c}_i^\top x + u_i^\top x$ and $\bar{c}_i \in \mathbb{R}^n$ for all $i \in I$. The corresponding single parametric problem can be expressed as

$$(LP_u^\beta)_{u \in \prod_{i=1}^m \mathcal{U}_i} \quad \text{V-min} \quad (\bar{c}_1^\top x + u_1^\top x, \dots, \bar{c}_m^\top x + u_m^\top x) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \forall (v_j, \gamma_j) \in \mathcal{V}_j, j \in J,$$

and the corresponding semi-robust counterpart reads

$$(\widetilde{LP}) \quad \text{V-min} \quad (\bar{c}_1^\top x, \dots, \bar{c}_m^\top x) \\ \text{s.t.} \quad \bar{a}_j^\top x + \bar{b}_j + v_j^\top x + \gamma_j \leq 0, \forall (v_j, \gamma_j) \in \mathcal{V}_j, j \in J.$$

As before, we assume that the uncertainty sets are $\mathcal{U}_i = \beta \mathcal{W}_i \subseteq \mathbb{R}^n$ with \mathcal{W}_i convex compact sets with $0_n \in \text{int } \mathcal{W}_i$, $\beta \geq 0$ and $\mathcal{V}_j \subseteq \mathbb{R}^{n+1}$ are all convex and compact. We remark that, in the special case of ball uncertainty sets, that is, $\mathcal{W}_i = \mathbb{B}_n$ for all $i \in I$, this model problem has been examined in [17].

Below, we obtain a sufficient condition for existence of highly robust weakly efficient solutions for linear multi-objective programs.

Corollary 13 (Existence of solutions: multi-objective linear programs) *Let $\mathcal{W}_i, i \in I$, be compact convex sets with $0_n \in \text{int } \mathcal{W}_i$ such that $\mathcal{W}_i \subseteq \rho \mathbb{B}_n$ for some $\rho > 0$. Consider the uncertain linear multi-objective program $(LP_{u,v}^\beta)$ and its semi-robust counterpart (\widetilde{LP}) . If there exist $\bar{x} \in X$ and $\lambda \in \Delta_m$ such that $-\sum_{i \in I} \lambda_i \bar{c}_i \in \text{int } N(X, \bar{x})$, then $\delta(\widetilde{LP}) \geq \frac{v_1^*}{\rho}$ with $v_1^* := \inf\{\|\sum_{i=1}^m \lambda_i \bar{c}_i + y\| : y \in \text{bd } N(X, \bar{x})\} > 0$ and a highly robust weakly efficient solution for $(LP_{u,v}^\beta)$ exists for all $\beta \in [0, \frac{v_1^*}{\rho}[$.*

Proof. The assumption $-\sum_{i \in I} \lambda_i \bar{c}_i \in \text{int } N(X, \bar{x})$ implies that

$$-\sum_{i=1}^m \lambda_i \bar{c}_i \in N(X, \bar{x}) \quad \text{and} \quad v_1^* := \inf\{\|\sum_{i=1}^m \lambda_i \bar{c}_i + y\| : y \in \text{bd } N(X, \bar{x})\} > 0. \quad (23)$$

So, the conclusion follows by applying Corollary 12 with $\bar{A}_i = 0_{n \times n}$. \square

The next straightforward consequence of Corollary 13 extends [17, Theorem 15] by dropping the assumption that X is a polytope and allowing the uncertainty set to be more general.

Corollary 14 Let $\mathcal{W}_i, i \in I$, be compact convex sets with $0_n \in \text{int } \mathcal{W}_i$ such that $\mathcal{W}_i \subseteq \rho \mathbb{B}_n$ for some $\rho > 0$. Consider the uncertain linear multi-objective program $(LP_{u,v}^\beta)$ and its semi-robust counterpart (\widetilde{LP}) . If there exists an index $i \in I$ and a corresponding extreme point \bar{a} of X such that $-\bar{c}_i \in \text{int } N(X, \bar{a})$, then $\delta(\widetilde{LP}) \geq \frac{v_1^*}{\rho}$ with $v_1^* := \inf \{\|\bar{c}_i + y\| : y \in \text{bd } N(X, \bar{a})\} = \text{dist}(-\bar{c}_i, \text{bd } N(X, \bar{a})) > 0$.

The above lower bound for $\delta(\widetilde{LP})$ was derived in the constructive proof of [17, Theorem 15] under the additional assumption that the robust feasible set X is a non-singleton polytope and the uncertainty sets are balls. The next example illustrates how one could make use of our Corollary 4 to determine the exact value of $\delta(\widetilde{LP})$, which explains the practical advantages of Corollary 4 instead of [17, Theorem 15].

Example 15 Consider the following linear uncertain multi-objective program which was examined in [17, Example 16]:

$$(LP_u)_{u \in \mathcal{U}} \quad \text{V-min}_{x \in X} \quad (\bar{c}_1^\top x + u_1^\top x, \bar{c}_2^\top x + u_2^\top x)$$

whose robust feasible set is $X = [-1, 1]^2$, and the uncertainty parameters $u_i, i = 1, 2$, belong to $\beta \mathbb{B}_2$. The vectors $\bar{c}_i, i = 1, 2$, are the (nominal) objective data given as below. The semi-robust counterpart problem (6) reads

$$(\widetilde{LP}) \quad \text{V-min}_{x \in X} \quad (\bar{c}_1^\top x, \bar{c}_2^\top x).$$

The candidates to be sharp efficient solution for (\widetilde{LP}) are the extreme points of X , namely, $a_1 = (1, 1), a_2 = (-1, 1), a_3 = (-1, -1)$ and $a_4 = (1, -1)$. We study the following two cases:

- (i) Let $\bar{c}_1 = (-2, -1)$ and $\bar{c}_2 = (-1, 1)$. In this case, a_1 is sharp efficient solution for (\widetilde{LP}) and so, $\delta(\widetilde{LP}) > 0$. Moreover, from Corollary 4,

$$\begin{aligned} \delta(\widetilde{LP}) &= \kappa(a_1, \widetilde{LP}) = \inf \left\{ \frac{\max\{3-2x_1-x_2, -x_1+x_2\}}{\|x-a_1\|} : x \in [-1, 1]^2 \setminus \{a_1\} \right\} \\ &= \inf \left\{ \frac{3-2x_1-x_2}{\|x-a_1\|} : x \in [-1, 1]^2 \setminus \{a_1\} \right\}, \end{aligned}$$

so that

$$\begin{aligned} \delta(\widetilde{LP})^2 &= \inf \left\{ \frac{(3-2x_1-x_2)^2}{(x_1-1)^2+(x_2-1)^2} : x \in [-1, 1]^2 \setminus \{a_1\} \right\} \\ &= \sup \left\{ \mu \in \mathbb{R}_+ : (3-2x_1-x_2)^2 \geq \mu [(x_1-1)^2 + (x_2-1)^2], \forall x \in [-1, 1]^2 \right\}. \end{aligned}$$

Taking $x = (1, 0)$, the first equality in the preceding relation gives us that $\delta(\widetilde{LP}) \leq 1$. We conclude that $\delta(\widetilde{LP}) = 1$ by observing that

$$\min_{x \in X} \left\{ (3-2x_1-x_2)^2 - [(x_1-1)^2 + (x_2-1)^2] \right\} = 0.$$

We note that [17, Example 16] only concludes that $\delta(\widetilde{LP}) \geq 1$.

(ii) Let $\bar{c}_1 = (1, 0)$ and $\bar{c}_2 = (-1, 0)$. None of the points a_i , $i = 1, \dots, 4$, is sharp efficient solution for (\tilde{LP}) . Consequently, Corollary 4 implies that $\delta(\tilde{LP}) = 0$.

We conclude this section by showing how our approach can be used to obtain conditions, guaranteeing the existence of highly robust solutions for portfolio optimization problems. So, consider the simple multi-objective portfolio optimization problem mentioned in the introduction

$$(P_{(r,A)}) \quad \text{V-min}_{x \in X} \quad (-r^\top x, x^\top Ax)$$

where the robust feasible set X is given by

$$X := \left\{ x \in \mathbb{R}^n : \begin{array}{l} a_j^\top x + b_j \leq 0, \forall (a_j, b_j) \in \mathcal{V}_j, \forall j \in J, \\ \sum_{i=1}^n x_i \leq C, x_i \geq 0, i = 1, \dots, n \end{array} \right\}$$

Recall that the spectral norm for an $(n \times n)$ matrix C is given by $\|C\|_{\text{spec}} = \sqrt{\lambda_{\max}(C^\top C)}$. Let \mathcal{V} be the matrix spectral norm uncertainty set given by $\mathcal{V} = \{C \succeq 0 : \|C\|_{\text{spec}} \leq 1\}$, where $C \succeq 0$ means C is positive semi-definite. Let $\bar{r} \in \mathbb{R}^n$ and $\bar{A} \succeq 0$. For each fixed $\beta \geq 0$, the corresponding single parametric problem has the form

$$(P_{u,C}^\beta)_{u \in \beta \mathbb{B}_n, C \in \beta \mathcal{V}} \quad \begin{array}{l} \text{V-min} \quad (-\bar{r}^\top x + u^\top x, x^\top (\bar{A} + C)x) \\ \text{s.t.} \quad x \in X, \end{array}$$

and the semi-robust counterpart becomes

$$(\tilde{P}) \quad \begin{array}{l} \text{V-min} \quad (-\bar{r}^\top x, x^\top \bar{A}x) \\ \text{s.t.} \quad x \in X. \end{array}$$

Recall that the set of all sharp efficient solutions of (\tilde{P}) is denoted by $\mathcal{S}(\tilde{P})$ and that the corresponding sharpness modulus of x with respect to (\tilde{P}) is denoted by $\kappa(x, \tilde{P})$. As noted before, for any $x \in \mathcal{S}(\tilde{P})$, one has $\kappa(x, \tilde{P}) > 0$.

We will see now that a suitable modification of the argument in Theorem 3 leads to a lower estimate of the radius of highly robust efficiency and a simple sufficient condition ensuring the non-emptiness of the highly robust (weakly) efficient solution set.

Proposition 16 *If $\bar{x} \in \mathcal{S}(\tilde{P})$ then $\delta(\tilde{P}) \geq \frac{\kappa(\bar{x}, \tilde{P})}{\max\{2\|\bar{x}\|, 1\}} > 0$. In particular, $X_h^\beta \neq \emptyset$ for all $0 < \beta < \frac{\kappa(\bar{x}, \tilde{P})}{\max\{2\|\bar{x}\|, 1\}}$.*

Proof. Let $\bar{x} \in \mathcal{S}(\tilde{P})$. Let $\beta > 0$ be such that $\beta < \frac{\kappa(\bar{x}, \tilde{P})}{\max\{2\|\bar{x}\|, 1\}}$. Take $k > 0$ be such that $\beta \cdot \max\{2\|\bar{x}\|, 1\} < k < \kappa(\bar{x}, \tilde{P})$. Let $\bar{f}_1(x) = -\bar{r}^\top x$ and $\bar{f}_2(x) = x^\top \bar{A}x$. Then, from the definition of the sharpness modulus of (\tilde{P}) (see (7)), one has

$$\max\{\bar{f}_1(x) - \bar{f}_1(\bar{x}), \bar{f}_2(x) - \bar{f}_2(\bar{x})\} \geq k \|x - \bar{x}\|, \quad \text{for all } x \in \mathbb{R}^n.$$

So, for all $u \in \beta \mathbb{B}_n$ and $C \in \beta \mathcal{V}$,

$$\begin{aligned} & \max\{\bar{f}_1(x) + u^\top x - (\bar{f}_1(\bar{x}) + u^\top \bar{x}), \bar{f}_2(x) + x^\top Cx - (\bar{f}_2(\bar{x}) + \bar{x}^\top C\bar{x})\} \\ & \geq \max\{\bar{f}_1(x) - \bar{f}_1(\bar{x}) + u^\top x - u^\top \bar{x}, \bar{f}_2(x) - \bar{f}_2(\bar{x}) + 2(C\bar{x})^\top (x - \bar{x})\} \\ & \geq \max\{\bar{f}_1(x) - \bar{f}_1(\bar{x}) - \|u\| \|x - \bar{x}\|, \bar{f}_2(x) - \bar{f}_2(\bar{x}) - 2\|C\bar{x}\| \|x - \bar{x}\|\}, \end{aligned}$$

where the first inequality follows from the positive semi-definiteness of C and the second inequality follows from the Cauchy–Schwartz inequality. As $C \in \beta\mathcal{V}$,

$$\|C\bar{x}\|^2 = \bar{x}^\top (C^\top C) \bar{x} \leq \lambda_{\max}(C) \|\bar{x}\|^2 \leq \beta^2 \|\bar{x}\|^2.$$

Now, for all $u \in \beta\mathbb{B}_n$ and for all $C \in \beta\mathcal{V}$,

$$\begin{aligned} & \max\{\bar{f}_1(x) + u^\top x - (\bar{f}_1(\bar{x}) + u^\top \bar{x}), \bar{f}_2(x) + x^\top Cx - (\bar{f}_2(\bar{x}) + \bar{x}^\top C\bar{x})\} \\ & \geq \max\{\bar{f}_1(x) - \bar{f}_1(\bar{x}), \bar{f}_2(x) - \bar{f}_2(\bar{x})\} - \beta \max\{2\|\bar{x}\|, 1\} \|x - \bar{x}\| \\ & \geq (k - \beta \max\{2\|\bar{x}\|, 1\}) \|x - \bar{x}\|. \end{aligned}$$

From our choice of k , we see that, for all $u \in \beta\mathbb{B}_n$ and for all $C \in \beta\mathcal{V}$, \bar{x} is a sharp efficient solution for $(P_{u,C}^\beta)$ (and so, is, in particular, a weakly efficient solution for $(P_{u,C}^\beta)$). This means that \bar{x} is a highly robust weakly efficient solution for $(P_{u,C}^\beta)_{u \in \beta\mathbb{B}_n, C \in \beta\mathcal{V}}$. Thus, $\delta(\tilde{P}) \geq \beta$ for all $\beta < \frac{\kappa(\bar{x}, \tilde{P})}{\max\{2\|\bar{x}\|, 1\}}$. This implies that

$$\delta(\tilde{P}) \geq \frac{\kappa(\bar{x}, \tilde{P})}{\max\{2\|\bar{x}\|, 1\}}.$$

Thus, the conclusion follows. \square

We now provide two examples to illustrate how one could estimate the radius of highly robust solutions for uncertain multi-objective optimization problems, where the uncertainty sets are generated as in the artificial portfolio problem in [4, Sec. 4].

Example 17 Consider the uncertain bi-objective optimization problem

$$(LP^\beta) \quad \text{V-} \min_{x \in \Delta_n} (-r^\top x, -w^\top x)$$

where the uncertain vectors r and w belong to the uncertainty sets $\bar{r} + \beta\mathcal{W}_1$ and $\bar{r} + \beta\mathcal{W}_2$, respectively. As in the example of [4, Sec. 4], we assume that $\mathcal{W}_1 = \prod_{k=1}^n [-\sigma_k, \sigma_k]$ (a box),

$\mathcal{W}_2 = \{u \in \mathbb{R}^n : \sum_{k=1}^n \sigma_k^{-2} u_k^2 \leq \theta^2\}$ (an ellipsoid whose size depends on the positive constant

θ), $n = 150$, $\bar{r}_k = 1.15 + \left(\frac{0.05}{150}\right)k$ and $\sigma_k = \left(\frac{0.05}{3}\right) \sqrt{\frac{151k}{75}}$ for $k = 1, \dots, 150$. Clearly, the finite sequences \bar{r}_k and σ_k are increasing. Denoting by $\{e_1, \dots, e_n\}$ the canonical basis of \mathbb{R}^n , it is not hard to verify that e_n is a weakly efficient solution for the problem

$$(\tilde{LP}) \quad \text{V-} \min_{x \in \Delta_n} (-\bar{r}^\top x, -\bar{r}^\top x)$$

and an extreme point of the feasible set $X = \Delta_n = \text{conv}\{e_1, \dots, e_n\}$. Note that

$$\begin{aligned} D(X, e_n) &= \text{cone}\{e_k - e_n : k = 1, \dots, n-1\}, \\ N(X, e_n) &= \{x \in \mathbb{R}^n : x_n \geq x_k, k = 1, \dots, n-1\}, \end{aligned}$$

and

$$\bar{r} \in \text{int } N(X, e_n) = \{x \in \mathbb{R}^n : x_n > x_k, k = 1, \dots, n-1\}.$$

For $k = 1, \dots, n-1$, the distance from \bar{r} to the hyperplane $x_k - x_n = 0$ is $\frac{|\bar{r}_k - \bar{r}_n|}{\sqrt{2}} = \frac{0.05(n-k)}{150\sqrt{2}}$, so that

$$\delta_0 := \text{dist}(\bar{r}, \text{bd } N(X, e_n)) = \min \left\{ \frac{0.05(n-k)}{150\sqrt{2}} : k = 1, \dots, n-1 \right\} = \frac{0.05}{150\sqrt{2}}.$$

Since $\mathcal{W}_1 \subset \sigma_{150}\sqrt{2}\mathbb{B}_{150}$ and $\mathcal{W}_2 \subset \sigma_{150}\theta\mathbb{B}_{150}$, one can apply Corollary 14, with $v_1^* = \delta_0$ and $\rho = \sigma_{150} \max\{\sqrt{2}, \theta\}$, to conclude that the radius of highly robust weak efficiency $\delta(\bar{L}P)$ is at least $\frac{v_1^*}{\rho} = \frac{1}{100\sqrt{151} \max\{\sqrt{2}, \theta\}}$.

Example 18 Consider now the uncertain bi-objective optimization problem

$$(QP^\beta) \quad V\text{-}\min_{x \in \Delta_n} (-r^\top x, x^\top A x)$$

where the vector r and the matrix A are uncertain, and they belong to the following uncertainty sets respectively:

$$r \in \bar{r} + \beta\mathbb{B}_n, \text{ and } A \in \bar{A} + \{C : C \succeq 0, \|C\|_{\text{spec}} \leq \beta\},$$

where $\bar{A} \succeq 0$ is a given matrix while n , e_k and \bar{r}_k are defined as in Example 17. Denote by $f_1(x) = -\bar{r}^\top x$ and $f_2(x) = x^\top \bar{A} x$ the objective functions of the problem (\widetilde{QP}) . Direct verification shows that, for all $x \in X = \Delta_n$,

$$\begin{aligned} f_1(x) - f_1(e_n) &= -\sum_{k=1}^n \bar{r}_k x_k + \bar{r}_n = -\sum_{k=1}^{n-1} \bar{r}_k x_k + \bar{r}_n(1 - x_n) \\ &\geq -\bar{r}_{n-1} \sum_{k=1}^{n-1} x_k + \bar{r}_n(1 - x_n) = (\bar{r}_n - \bar{r}_{n-1})(1 - x_n) \\ &= \frac{\bar{r}_n - \bar{r}_{n-1}}{2} \|x - e_n\|_1 \geq \frac{\bar{r}_n - \bar{r}_{n-1}}{2} \|x - e_n\|, \end{aligned}$$

where the first inequality follows from the fact that \bar{r}_k is a finite increasing sequence, the third equality follows from $x \in X$, the fourth equality holds because $x \in X$ and

$$\|x - e_n\|_1 = \sum_{k=1}^{n-1} |x_k| + |x_n - 1| = \sum_{k=1}^{n-1} x_k + (1 - x_n) = 2(1 - x_n),$$

and the last inequality follows since $\|\cdot\|_1 \geq \|\cdot\|$. This shows that e_n is a sharp efficient solution for the problem (\widetilde{QP}) and

$$\kappa(e_n, \widetilde{QP}) \geq \frac{\bar{r}_n - \bar{r}_{n-1}}{2}.$$

Hence, Proposition 16 shows that the radius of highly robust weak efficiency is at least $\frac{\bar{r}_n - \bar{r}_{n-1}}{4} = \frac{1}{12000}$. Since only f_1 is relevant in the above computations, the latter lower bound for the radius of highly robust weak efficiency also applies to the problem in Example 17, and it is tighter than the one obtained there whenever $\theta \geq \frac{120}{\sqrt{151}}$.

6 Conclusions and further research

In this paper we have established bounds and an exact formula (see Theorem 3 and Corollary 4) for the radius of highly robust efficiency. The exact formula provides a certificate for the existence of highly robust weakly efficient solutions in the case where the objective functions are affected by the commonly used ball uncertainty.

Such a formula involves the moduli of the sharp solutions of the semi-robust counterpart (\tilde{P}) , which is an unexpected finding, and can be applied when at least one sharp solution of (\tilde{P}) is available since its modulus is a lower bound for the radius of highly robustness.

On the other hand, finding formulas expressing the radius of highly robustness in terms of the data is a challenging open problem even in the case of convex quadratic and linear multi-objective programs. However, Theorem 9 to these particular problems provides relatively simple formulas under mild conditions (Corollaries 12 and 13). Moreover, Corollary 14 is a significant improvement of [17, Theorem 15].

The results in this paper warrant further research in several directions. In particular, it would be of great interest to examine how the results of this paper can be extended by replacing the weak efficient solutions (whose characterization via scalarization has been used in many proofs) with efficient, properly efficient and strongly efficient solutions.

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